Indian Statistical Institute, Bangalore Centre
B.Math (Hons.) II Year, First Semester
Mid-Sem Examination
Algebra III

Time: 9.30AM-1.00PM September 17, 2012 Instructor: Bhaskar Bagchi
Maximum Marks : 100

Note : There are five questions; each carries 20 marks.

1. (a) Let R be a ring except that addition and multiplication in R are not assumed to be commutative. Show that addition must be commutative. (Hint : for $a, b, c, d \in R$, look at $(a+b)(c+d)$).

   (b) Let R be a ring and let $e \in R$ satisfy $e^2 = e$. Show that there is a subset S of R, closed under addition and multiplication, such that S is a ring with the induced operations and e is the multiplicative identity of S.

2. (a) Define the prime and irreducible elements of an integral domain and show that the primes are always irreducible. Given an example (with proof) of an irreducible element which is not prime.

   (b) Find all the irreducible element of the ring A of algebraic integers. Show that A is not Noetherian.

3. (a) Find all the maximal ideals in the ring $\mathbb{C}[X_1, \ldots, X_n]$. (You may assume the fundamental theorem of algebra without proof.)

   (b) Deduce that there is a natural bijection between the varieties in $\mathbb{C}^n$ and the radical ideals in this ring.

4. (a) Find all the prime elements in the ring of Gaussian integers. (You may assume without proof that for primes $p > 0$ in Z, -1 is a square in Z/pZ iff $p = 2$ or $p \equiv 1 (\mod 4)$).

   (b) List all the Gaussian primes $a + bi$ with $a^2 + b^2 \leq 20$.

5. (a) Define the primitive elements of the ring $Z[X]$. Show that the product of two primitive elements is primitive.

   (b) Deduce that, if $f, g \in Z[X]$ are primitive and f divides g in $Q[X]$, then f divides g in $Z[X]$.